课程：8.3 P173

题号	一	二	三	四	五	六	七	八	九	十	总分
得分											
阅卷人											

考生注意：答题时应注明题号，并按题号顺序答题。

题号 P173 答题内容 理学院 廖国勇

一 1 设 $z=\frac{y}{x}$，而 $x=e^t$，$y=1-\cos t$，则 $\frac{dz}{dt}=$ ___

解 方法一，$z=\frac{y}{x}=(1-\cos t)e^{-t}$

$$\frac{dz}{dt}=\sin t\cdot e^{-t}+(1-\cos t)e^{-t}\cdot(-1)$$

$$=e^{-t}(\sin t+\cos t)-e^{-t}$$

方法二，z 〈 $y-t$，$x-t$

$$\frac{dz}{dt}=\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}$$

$$=\frac{1}{x}\sin t+-\frac{y}{x^2}e^t$$

$$=e^{-t}\sin t-\frac{1-\cos t}{e^{2t}}\cdot e^t$$

$$=e^{-t}(\sin t+\cos t)-e^{-t}$$

2 设 $u=e^{x^2+y^2+z^2}$，而 $z=x^2\sin y$，则 $\frac{\partial u}{\partial x}=$ ___

解 $\frac{\partial u}{\partial x}=e^{x^2+y^2+z^2}\cdot(2x+2z\cdot 2x\sin y)$

$$=e^{x^2+y^2+x^2\sin y}(2x+4x^3\sin^2 y)$$

答案中 z 不用 $x^2\sin y$ 代替也可以.

设 $V=x^2+y^2+z^2$，则 $u-V$ 〈 x,y,z

$$\frac{\partial u}{\partial x}=\frac{du}{dV}\cdot\frac{\partial V}{\partial x}+\frac{du}{dV}\cdot\frac{\partial V}{\partial z}\cdot\frac{\partial z}{\partial x}$$ z 〈 x,y

一、3 设函数 $y=y(x)$ 由方程 $\sin y+e^x=xy^2$ 确定，则 $\frac{dy}{dx}=$ ____

解 方法一，公式法

设 $F(x,y)=\sin y+e^x-xy^2$

$\frac{\partial F}{\partial x}=e^x-y^2$ $\quad \frac{\partial F}{\partial y}=\cos y-2xy$

$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{e^x-y^2}{\cos y-2xy}$

方法二，方程两边同时对 x 求导，y 看成 $y(x)$

$\cos y\cdot\frac{dy}{dx}+e^x=y^2+x\cdot 2y\cdot\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{y^2-e^x}{\cos y-2xy}$

4 设函数 $z=z(x,y)$ 由方程 $\frac{x}{z}=\ln\frac{z}{y}$ 确定，则 $\frac{\partial z}{\partial x}=$ ____

解 方法一，公式法

设 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}$，则 $\frac{\partial F}{\partial x}=\frac{1}{z}$

$\frac{\partial F}{\partial z}=-\frac{x}{z^2}-\frac{1}{\frac{z}{y}}\cdot\frac{1}{y}=-\frac{z+x}{z^2}$

$\frac{\partial z}{\partial x}=-\left(\frac{\partial F}{\partial x}/\frac{\partial F}{\partial z}\right)$

$=-\left(\frac{1}{z}/(-1)\frac{z+x}{z^2}\right)=\frac{z}{z+x}$

方法二，方程两边同时对 x 求偏导，把 z 看成 $z(x,y)$

左边函数 $\begin{cases}x\\ z<\begin{matrix}x\\ y\end{matrix}\end{cases}$ 右边函数 $-z\begin{cases}x\\ y\end{cases}$

$\frac{1}{z}+\left(-\frac{z}{x^2}\right)\cdot\frac{\partial z}{\partial x}=\frac{1}{\frac{z}{y}}\cdot\frac{1}{y}\cdot\frac{\partial z}{\partial x}=\frac{1}{z}\frac{\partial z}{\partial x}$

$\Rightarrow \frac{\partial z}{\partial x}=\frac{\frac{1}{z}}{\frac{1}{z}+\frac{z^2}{x}}=\frac{z+x}{z}$

1 设 $z=\arctan(xy)$，且 $y=e^x$，求 $\frac{dz}{dx}$，$\frac{dz}{dy}$

解 $z \begin{cases} xy \begin{cases} x \\ y \end{cases} \\ y \end{cases}$

$$\frac{dz}{dx}=\frac{1}{1+x^2y^2}(y+x\cdot e^x)=\frac{y+xe^x}{1+x^2y^2}=\frac{e^x+xe^x}{1+x^2e^{2x}}$$

解 $z \begin{cases} xy \begin{cases} x \\ y \end{cases} \\ y \end{cases}$ $x=\ln y$

$$\frac{dz}{dy}=\frac{1}{1+x^2y^2}\cdot\left(y\cdot\frac{1}{y}+x\right)=\frac{1+x}{1+x^2y^2}$$

$$=\frac{1+\ln y}{1+y^2\ln^2 y}$$

2 设 $z=u^2\ln v$，且 $u=\frac{x}{y}$，$v=4x-3y$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$

解 $z \begin{cases} u \begin{cases} \frac{x}{y} \\ \frac{x}{y} \end{cases} \\ v \begin{cases} x \\ y \end{cases} \end{cases}$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=2u\ln v\cdot\frac{1}{y}+\frac{u^2}{v}\cdot 4$$

$$=\frac{2u\ln v}{y}+\frac{4u^2}{v}=\frac{2\cdot\frac{x}{y}\ln(4x-3y)}{y}+\frac{4\frac{x^2}{y^2}}{4x-3y}$$

$$=2x\ln(4x-3y)/y+4x^2/y^2(4x-3y)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=2u\ln v\cdot\left(-\frac{x}{y^2}\right)+\frac{u^2}{v}\cdot(-3)$$

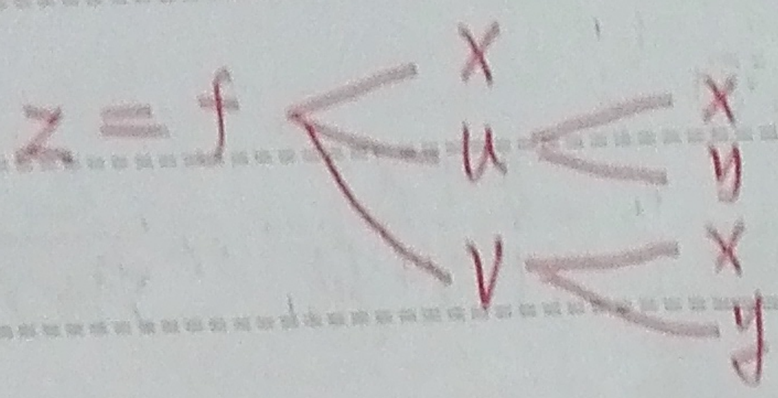
$$=-\frac{2xu\ln v}{y^2}-\frac{3u^2}{v}$$

$$=-\frac{2x\cdot\frac{x}{y}\ln(4x-3y)}{y^2}-\frac{3\cdot\frac{x^2}{y^2}}{4x-3y}$$

$$=-\frac{2x^2\ln(4x-3y)}{y^3}-\frac{3x^2}{(4x-3y)y^2}$$

$z=f(x,u,v)$，且 $u=2x+y$，$v=xy$，其中 f 具有一阶连续偏导，求 dz

分析 z 的结构图



解 $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$

$=f_1'+f_2'\cdot 2+f_3'\cdot y$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_2'\cdot 1+f_3'\cdot x$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(f_1'+2f_2'+yf_3')dx+(f_2'+xf_3')dy$

设函数 $z=z(x,y)$ 由方程 $xz=\sin y+f(xy,z+y)$ 确定，其中 f 具有一阶连续偏导，求 dz

解 设 $F(x,y,z)=xz-\sin y-f(xy,z+y)$

$F_x'=z-yf_1'$　　$F_y'=-\cos y-xf_1'-f_2'$

$F_z'=x-f_2'$

$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{z-yf_1'}{f_2'-x}$　　$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=\frac{-\cos y-xf_1'-f_2'}{f_2'-x}$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$=\frac{z-yf_1'}{f_2'-x}dx+\frac{\cos y+xf_1'+f_2'}{x-f_2'}dy$

求 $\frac{\partial z}{\partial x}$ 也可以用两边（方程）对 x 和 y 求导得

以上解答是用公式法求 $\frac{\partial z}{\partial x}$

5 设函数 $z=z(x,y)$ 由方程 $z^5-xz^4+yz^3=1$ 确定，求 $\frac{\partial z}{\partial y}\Big|_{\substack{x=0\\y=0}}$

解 方程两边同时对 x 求偏导

$$5z^4\cdot\frac{\partial z}{\partial x}-z^4+x\cdot 4z^3\cdot\frac{\partial z}{\partial x}+y3z^2\cdot\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{-z^2}{4xz-5z^2-3y}$$

方程两边同时对 y 求偏导

$$5z^4\cdot\frac{\partial z}{\partial y}-x4z^3\frac{\partial z}{\partial y}+z^3+y\cdot 3z^2\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}=\frac{z}{4xz-5z^2+3y}$$

把 $x=0$，$y=0$ 代入方程得 $z=1$，于是

$$\frac{\partial z}{\partial x}\Big|_{\substack{x=0\\y=0}}=\frac{1}{5}\qquad \frac{\partial z}{\partial y}\Big|_{\substack{x=0\\y=0}}=-\frac{1}{5}$$

6 设 $u=f(x,y,z)$ 具有一阶连续偏导，$y=y(x)$ 及 $z=z(x)$ 分别由方程 $e^{xy}-xy=2$ 及 $e^x=\int_0^{x-z}\frac{\sin t}{t}dt$ 确定，求 $\frac{du}{dx}$

解 设 $F(x,y)=e^{xy}-xy-2$ $F_x'=ye^{xy}-y$ $F_y'=xe^{xy}-x$

$$\frac{dy}{dx}=-\frac{Fx'}{Fy'}=-\frac{ye^{xy}-y}{xe^{xy}-x}=-\frac{y}{x}$$

设 $G(x,z)=e^x-\int_0^{x-z}\frac{\sin t}{t}dt$

$$Gx'=e^x-\frac{\sin(x-z)}{x-z}\qquad Gz'=-\frac{\sin(x-z)}{x-z}\cdot(-1)=\frac{\sin(x-z)}{x-z}$$

$$\frac{dz}{dx}=-\frac{Gx'}{Gz'}=1+\frac{(z-x)e^x}{\sin(x-z)}$$

$$\frac{du}{dx}=f_1'+f_2'\frac{dy}{dx}+f_3'\frac{dz}{dx}$$

$$=f_1'+f_2'\left(-\frac{y}{x}\right)+f_3'\left[1+\frac{(z-x)e^x}{\sin(x-z)}\right]$$

f_1' 表示 f 对其第 1 个变元求偏导，其余类似

二、设 $z=x^3f(xy,\frac{y}{x})$，$f(u,v)$ 具有二阶连续偏导，$\frac{\partial^2 z}{\partial y^2}$，$\frac{\partial^2 z}{\partial x\partial y}$

解 $\frac{\partial z}{\partial y}=x^3\left[f_1'\cdot x+f_2'\cdot\frac{1}{x}\right]=x^4f_1'+x^2f_2'$

$\frac{\partial^2 z}{\partial y^2}=x^4\left(f_{11}''\cdot x+f_{12}''\cdot\frac{1}{x}\right)+x^2\left(f_{21}''\cdot x+f_{22}''\cdot\frac{1}{x}\right)$

$=x^5f_{11}''+x^3f_{12}''+x^3f_{21}''+xf_{22}''$

$=x^5f_{11}''+2x^3f_{12}''+xf_{22}''$

$\frac{\partial^2 z}{\partial x\partial y}=4x^3f_1'+x^4\cdot\left[f_{11}''\cdot y+f_{12}''\left(-\frac{y}{x^2}\right)\right]+12x^2f_2'+x^2\left[f_{21}''y-\frac{y}{x^2}f_{22}''\right]$

$=4x^3f_1'+2xf_2'+x^4yf_{11}''+(-y)f_{22}''$

8 设函数 $y=y(x)$，$z=z(x)$ 由方程组 $\begin{cases}z=xf(x+y)\\F(x,y,z)=0\end{cases}$ 确定，求 $\frac{dz}{dx}$

解 方程组中每个方程两边同时对 x 求导

$$\begin{cases}\frac{dz}{dx}=f+xf'\cdot\left(1+\frac{dy}{dx}\right)\\F_1'+F_2'\cdot\frac{dy}{dx}+F_3'\frac{dz}{dx}=0\end{cases}$$

即 $$\begin{cases}xf'\frac{dy}{dx}-\frac{dz}{dx}=-f-xf'\\F_2'\frac{dy}{dx}+F_3'\frac{dz}{dx}=-F_1'\end{cases}$$

$$\frac{dz}{dx}=\frac{\begin{vmatrix}xf' & -f-xf'\\F_2' & -F_1'\end{vmatrix}}{\begin{vmatrix}xf' & -1\\F_2' & F_3'\end{vmatrix}}=\frac{-xf'F_1'+fF_2'+xf'F_2'}{xf'F_3'+F_2'}$$

$F_1'=F_x'\quad F_2'=F_y'\quad F_3'=F_z'$

二 9 设 $u=u(x,y)$，$v=v(x,y)$ 由方程组 $\begin{cases} xu-yv=0 \\ yu+xv=1 \end{cases}$ 确定，求 $\frac{\partial u}{\partial x}$，$\frac{\partial v}{\partial x}$ 及 $\frac{\partial u}{\partial y}$，$\frac{\partial v}{\partial y}$

解 方程组每个方程两边对 x 求导

$$\begin{cases} u+x\frac{\partial u}{\partial x}-y\frac{\partial v}{\partial x}=0 \\ y\frac{\partial u}{\partial x}+v+x\frac{\partial v}{\partial x}=0 \end{cases} \Rightarrow \frac{\partial u}{\partial x}=-\frac{xu+yv}{x^2+y^2} \quad \frac{\partial v}{\partial x}=\frac{-xv+yu}{x^2+y^2}$$

方程组每个方程两边对 y 求导

$$\begin{cases} x\frac{\partial u}{\partial y}-v-y\frac{\partial v}{\partial y}=0 \\ u+y\frac{\partial u}{\partial y}+x\frac{\partial v}{\partial y}=0 \end{cases} \Rightarrow \frac{\partial u}{\partial y}=-\frac{-vx+uy}{x^2+y^2} \quad \frac{\partial v}{\partial y}=-\frac{xu+yv}{x^2+y^2}$$

三 1 设函数 $z=f(x,y)$ 由方程 $F(x-y,y-z,z-x)=0$ 确定，$F(u,v,w)$ 具有连续偏导数，且 $F_v'-F_u'\neq 0$，证明：$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$

证明 ~~方程两边~~ 公式法

设 $g(x,y,z)=F(x-y,y-z,z-x)$

$$\frac{\partial g}{\partial x}=F_u'-F_w' \quad \frac{\partial g}{\partial y}=-F_u'+F_v' \quad \frac{\partial g}{\partial z}=-F_v'+F_w'$$

$$\frac{\partial z}{\partial x}=-\frac{\partial g/\partial x}{\partial g/\partial z}=-\frac{F_u'-F_w'}{-F_v'+F_w'}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}}=-\frac{-F_u'+F_v'}{-F_v'+F_w'}$$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=-\frac{F_u'-F_w'}{-F_v'+F_w'}-\frac{-F_u'+F_v'}{-F_v'+F_w'}=1$$

三 2 P176

设函数 $z=f(x,y)$ 由方程 $\phi(x+\frac{z}{y}, y+\frac{z}{x})=0$ 确定，其中 $\phi(u,v)$ 具有连续偏导数，证明：$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$

证明 设 $g(x,y,z)=\phi(x+\frac{z}{y}, y+\frac{z}{x})$，则

$$\frac{\partial g}{\partial x}=\phi_u'+\phi_v'$$

$$\frac{\partial g}{\partial x}=\phi_u'(1+\frac{1}{y}\cdot\frac{\partial z}{\partial x})$$

$$\frac{\partial g}{\partial x}=\phi_u'+\phi_v'\cdot(-\frac{z}{x^2}) \qquad \frac{\partial g}{\partial y}=\phi_u'\cdot(-\frac{z}{y^2})+\phi_v'$$

$$\frac{\partial g}{\partial z}=\phi_u'\frac{1}{y}+\phi_v'\cdot\frac{1}{x}$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}}=-\frac{\phi_u'+\phi_v'(-\frac{z}{x^2})}{\phi_u'\frac{1}{y}+\phi_v'\frac{1}{x}}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}}=-\frac{\phi_u'(-\frac{z}{y^2})+\phi_v'}{\phi_u'\frac{1}{y}+\phi_v'\cdot\frac{1}{x}}$$

$$x\cdot\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{x\phi_u'-\frac{z}{x}\phi_v'+y\phi_v'-\frac{z}{y}\phi_u'}{\phi_u'\frac{1}{y}+\phi_v'\frac{1}{x}}$$

$$=z-xy$$

9 讨论函数 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在点(0,0)处的可微性

解：$f_x'(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0$

同理 $f_y'(0,0)=0$

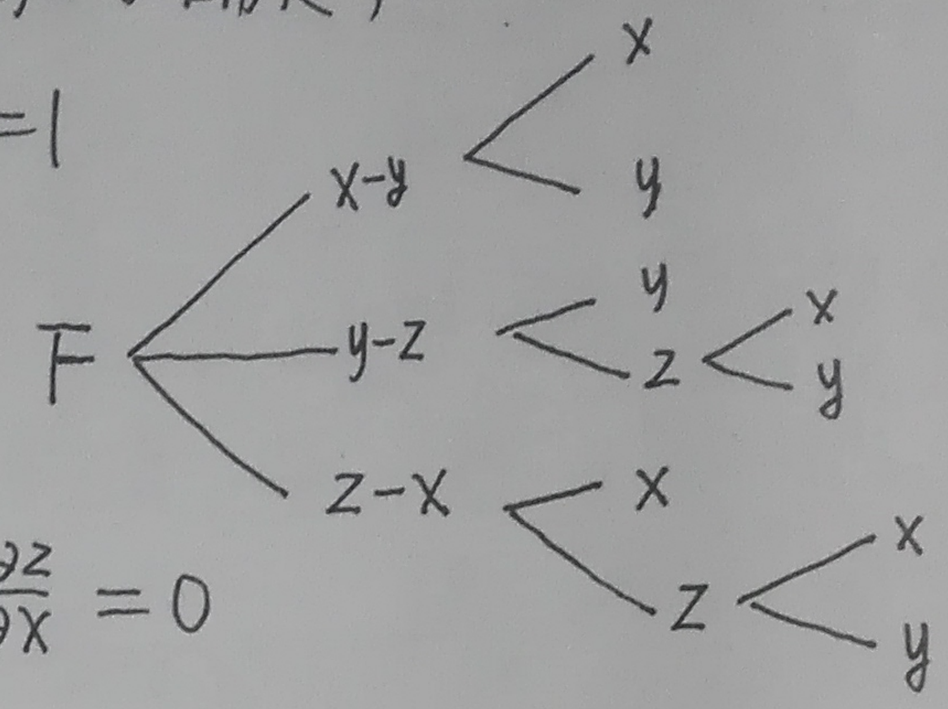
于是，$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(0+\Delta x,0+\Delta y)-f(0,0)-f_x'(0,0)\Delta x-f_y'(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$

$=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\dfrac{\Delta x\cdot\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\Delta x\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}$ 不存在

所以，函数 $f(x,y)$ 在(0,0)处不可微。

同步练习 P176 三 1. 廖国勇 2016·4·12

1、设函数 $z=f(x,y)$ 由方程 $F(x-y, y-z, z-x)=0$ 确定，$F(u,v,w)$ 具有连续偏导数，且 $F_v' - F_w' \neq 0$，证明：$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$



证明：方程 $F(x-y, y-z, z-x)=0$

两边同时对 x 求导.

$$F_u' \cdot 1 + F_v' \cdot (-1) \cdot \frac{\partial z}{\partial x} + F_w' \cdot (-1) + F_w' \cdot 1 \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{F_u' - F_w'}{F_v' - F_w'}$$

方程 $F(x-y, y-z, z-x)=0$ 两边同时对 x 求导

$$F_u' \cdot (-1) + F_v' \cdot 1 + F_v' (-1) \cdot \frac{\partial z}{\partial y} + F_w' \cdot 1 \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{F_v' - F_u'}{F_v' - F_w'}$$

于是 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} = \frac{F_u' - F_w'}{F_v' - F_w'} + \frac{F_v' - F_u'}{F_v' - F_w'} = 1$

同步练习 P176 2　　廖国勇　2016.4.12

2 设函数 $z=f(x,y)$ 由方程 $\phi(x+\frac{z}{y}, y+\frac{z}{x})=0$ 确定，其中 $\phi(u,v)$ 具有连续偏导数，证明 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$

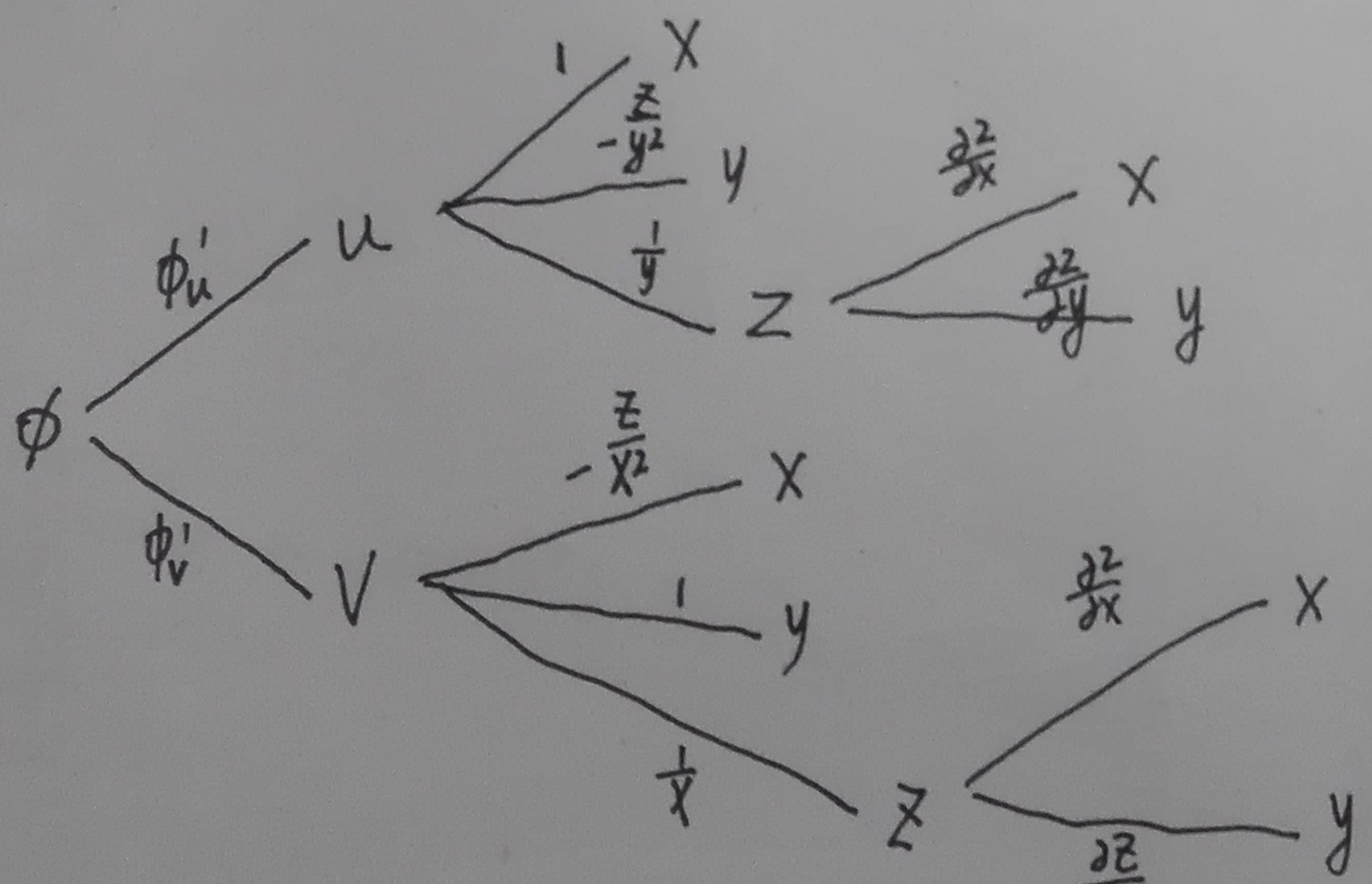
证明：方程 $\phi(x+\frac{z}{y}, y+\frac{z}{x})=0$ 两边同时对 x 求导

$$\phi_u' + \phi_u' \cdot \frac{1}{y} \cdot \frac{\partial z}{\partial x} + \phi_v' \cdot (-\frac{z}{x^2}) + \phi_v' \cdot \frac{1}{x} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{\frac{z}{x^2}\phi_v' - \phi_u'}{\frac{\phi_u'}{y} + \frac{\phi_v'}{x}} = \frac{yz\phi_v' - x^2y\phi_u'}{x^2\phi_u' + xy\phi_v'} \qquad (*)$$

同理可得：$\frac{\partial z}{\partial y} = \frac{xz\phi_u' - xy^2\phi_v'}{y^2\phi_v' + xy\phi_u'}$（不要计算，在(*)中 x 换 y，ϕ_u' 换 ϕ_v'；y 换 x，ϕ_v' 换 ϕ_u'）

于是，$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{yz\phi_v' - x^2y\phi_u'}{x\phi_u' + y\phi_v'} + \frac{xz\phi_u - xy^2\phi_v'}{y\phi_v' + x\phi_u'}$

$= z-xy$

如果想不通，撞墙三下就想通